Entropy for Black Holes in the Deformed Horava-Lifshitz Gravity

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We study the entropy of black holes in the deformed Horava-Lifshitz gravity with coupling constant λ. For $\lambda = 1$, the black hole resembles the Reissner-Norstrom black hole with a geometric parameter acting like the electric charge. Therefore, we obtain some differences in the entropy when comparing with the Schwarzschild black hole. Finally, we study the heat capacity and the thermodynamical stability of this solution.

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I. INTRODUCTION

Recently, Horava [1] proposed a non-relativistic renormalisable theory of gravity that reduces to Einstein's general relativity at large scales. This theory is named Horava-Lifshitz theory and has been studied in the literature for its applications to cosmology [2] and black holes [3]. However, this proposal introduces back a non-equality of space and time. Therefore, in this approach, space and time exhibit Lifshitz scale invariance $t \to l^z t$ and $x^i \to l x^i$ with $z \geq 1$. Moreover, the theory is not invariant under the full diffeomorphism group of General Relativity (GR), but rather under a subgroup of it. This fact is manifest using the standard ADM splitting.

However, the Horava-Lifshitz theory goes to standard GR if the coupling λ that controls the contribution of the trace of the extrinsic curvature has the specific value $\lambda = 1$. For generic values of λ, the theory does not exhibits the full 4D diffeomorphism invariance at large distances and it is possible to obtain deviations from GR. Therefore, it is interesting to confront this type of non-relativistic theory with experimental and observational data.

Using the (3 + 1)-dimensional ADM formalism, the general metric can be written as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{1}$$

where g_{ij}, N and N^i are the dynamical fields of scaling mass dimensions 0, 0, 2, respectively. The Einstein-Hilbert action can be expressed as

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} N \{ \left(K_{ij}K^{ij} - K^2 \right) + R - 2\Lambda \}, \tag{2}$$

where G is Newton's constant and the extrinsic curvature K_{ij} takes the form

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \tag{3}$$

with a dot denoting derivative with respect to t and covariant derivatives defined with respect to the spatial metric g_{ij}. On the other hand, the action of Horava-Lifshitz theory is given by [1]

$$\begin{aligned} S_{HL} \;=\; \int dt d^3x \sqrt{g} N \Bigg[& \frac{2}{\kappa^2} \left(K_{ij}K^{ij} - \lambda K^2 \right) + \frac{\kappa^2 \mu^2 (\tilde{\Lambda} R - 3\tilde{\Lambda}^2)}{8(1-3\lambda)} + \frac{\kappa^2 \mu^2 (1-4\lambda)}{32(1-3\lambda)} R^2 \\ & - \frac{\kappa^2 \mu^2}{8} R_{ij} R^{ij} + \frac{\kappa^2 \mu}{2\omega^2} \epsilon^{ijk} R_{il} \nabla_j R^l_k - \frac{\kappa^2}{2\omega^4} C_{ij} C^{ij} \Bigg], \end{aligned} \tag{4}$$

where κ^2, λ, ω are dimensionless constant parameters while μ and Λ are constant parameters with mass dimensions $[\mu] = 1$, $[\Lambda] = 2$. The object C_{ij} is called the Cotten tensor, defined by

$$C^{ij} = \epsilon^{ijk}\nabla_k R_l^j - \frac{1}{4}\epsilon^{ijk}\partial_k R. \tag{5}$$

Comparing the action to that of general relativity, one can see that the speed of light, Newton's constant and the cosmological constant are

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\tilde{\Lambda}}{1-3\lambda}} \tag{6}$$

$$G = \frac{\kappa^2 c}{32\pi} \tag{7}$$

$$\Lambda = \frac{3}{2}\tilde{\Lambda}. \tag{8}$$

Note that if $\lambda = 1$, the first two terms in (4) could be reduced to the Einstein's general relativity action (2). However, in Horava-Lifshitz theory, λ is a dynamical coupling constant, susceptible to quantum correction.

The static, spherically symmetric solutions have been found in [3]. Because of the prescence of a cosmological constant, solutions for $\lambda = 1$ are asymptotically AdS and have some interest because the AdS/CFT correspondence. These solutions have also been extended to general topological black holes[4], in which the 2-sphere that acts as horizon has been generalized to two dimensional constant curvature spaces.

II. HORAVA-LIFSHITZ BLACK HOLE

Now we will introduce the black hole solution in the limit of $\tilde{\Lambda} \to 0$ and its thermodynamic properties. Considering $N_i = 0$ (spherically symmetric solutions) in (1) we obtain the metric ansatz

$$ds^2 = -N^2(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d^2\Omega. \tag{9}$$

Using this line element and after the angular integration, the Lifshitz-Horava lagrangian reduces to

$$\tilde{\mathcal{L}}_1 = \frac{\kappa^2\mu^2 N}{8(1-3\lambda)\sqrt{f}}\left(\frac{\lambda-1}{2}f'^2 - \frac{2\lambda(f-1)}{r}f' + \frac{(2\lambda-1)(f-1)^2}{r^2} - 2\omega(1-f-rf')\right), \tag{10}$$

where

$$\omega = \frac{8\mu^2(3\lambda-1)}{\kappa^2}. \tag{11}$$

For $\lambda = 1$, we have $\omega = \frac{16\mu^2}{\kappa^2}$ and the functions f and N can be determined as [5]

$$N^2 = f(r) = 1 + \omega r^2 - \sqrt{r(\omega^2 r^3 + 4\omega M)}, \tag{12}$$

where M is an integration constant that will be related to the mass of the black hole. The condition $f(r_\pm) = 0$ defines the radii of the horizons

$$r_\pm = M\left[1 \pm \sqrt{1 - \frac{1}{2\omega M^2}}\right]. \tag{13}$$

This equation shows that

$$M^2 \geq \frac{1}{2\omega} \tag{14}$$

in order to have a black hole. The equality corresponds to the extremal black hole in which the degenerate horizon has a radius

$$r_e = M_e = \frac{1}{\sqrt{2\omega}}. \tag{15}$$

In order to compare with Schwarzschild's slution, we define a new parameter α as

$$\alpha = \frac{1}{2\omega}, \tag{16}$$

so the function f becomes

$$f(r) = \frac{2r^2 - 4Mr + 2\alpha}{r^2 + 2\alpha + \sqrt{r^4 + 8\alpha Mr}}. \tag{17}$$

Note that $f \to 2\left(1 - \frac{2M}{r}\right)$ as $\alpha \to 0$, i.e. that we recover Schwarzschild's solution when $\omega \to \infty$. Now, the horizon radii become

$$r_\pm = M \pm \sqrt{M^2 - \alpha}, \tag{18}$$

showing an incredible resemblance with the Reissner-Nordstrom solution in which the horizons are defined by $r_\pm = M \pm \sqrt{M^2 - Q^2}$, i.e. that the parameter α can be associated with the electric charge. In terms of α, the extremal black hole i characterized by the degenerate horizon

$$r_e = M_e = \sqrt{\alpha}. \tag{19}$$

Since this spacetime is spherically symmetric, the temperature of the black hole can be calculated as

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \left.\frac{\partial f(r)}{\partial r}\right|_{r=r_+}, \tag{20}$$

or using equation (17),

$$T = \frac{r_+^2 - \alpha}{4\pi(r_+^3 + 2\alpha r_+)}. \tag{21}$$

Note that this temperature becomes Schwarzschild's black hole temperature $T_s = \frac{1}{4\pi r_+}$ for $\alpha = 0$. Evenmore, in the extremal case, $r_+ = r_e = \sqrt{\alpha}$ the temperature vanishes.

III. ENTROPY

Using the condition $f\left(r_+\right) = 0$, the mass function is given by

$$M(r_+, \omega) = \frac{1 + 2\omega r_+^2}{4\omega r_+}, \tag{22}$$

or in terms of the parameter α,

$$M(r_+, \alpha) = \frac{\alpha + r_+^2}{2r_+} \tag{23}$$

In Einstein's general relativity, entropy of black hole is always given by one quarter of black hole horizon area, but in higher derivative gravities, in general, the area formula breaks down. Therefore, we will obtain the black hole entropy by using the first law of black hole thermodynamics, assuming that this black hole is a thermodynamical system and the first law keeps valid,

$$dM = TdS. \tag{24}$$

Note that we do not associate a thermodynamical character to the parameter α. Integrating this relation yields

$$S = \int \frac{dM}{T} + S_0, \tag{25}$$

where S_0 is an integration constant, which should be fixed by physical consideration. Since the mass of the black hole is a function of r_+ we can write

$$S = \int \frac{1}{T} \frac{\partial M}{\partial r_+} dr_+ + S_0. \tag{26}$$

Using the temperature (21) and the mass formula (23), we obtain the entropy

$$S = S_0 + \pi \left(r_+^2 + 4\alpha \ln(r_+) \right), \tag{27}$$

that is similar to the entropy obtained for topological black holes in [4]. Since we want that this entropy becomes one quarter of black hole horizon area for Schwarzschild's limit (i.e. $\alpha = 0$), we conclude that the integration constant is $S_0 = 0$, giving the entropy

$$S = \pi r_+^2 + 4\pi\alpha \ln(r_+). \tag{28}$$

In Figure 1 it is shown the behavior of the entropy as a function of the horizon radius in the case $\alpha = \frac{1}{2}$, i.e. $\omega = 1$ for the Horava-Lifshitz black hole as well as for Schwarzschild's solution. Note how the curves coincide for large r_+ but there is a significant difference near $r_+ = 0$, because the Horava-Lifshitz entropy diverges at this point. However, remember that the Horava-Lifshitz black hole exists for values of $M \geq \sqrt{\alpha} = \frac{1}{\sqrt{2\omega}}$, therefore, it can only have horizon radii satisfying $r_+ \geq r_e = M_e$, or $r_+ \geq \frac{1}{\sqrt{2\omega}}$. Thus, the radius $r_+ = 0$ is not allowed and there is no entropy divergence.

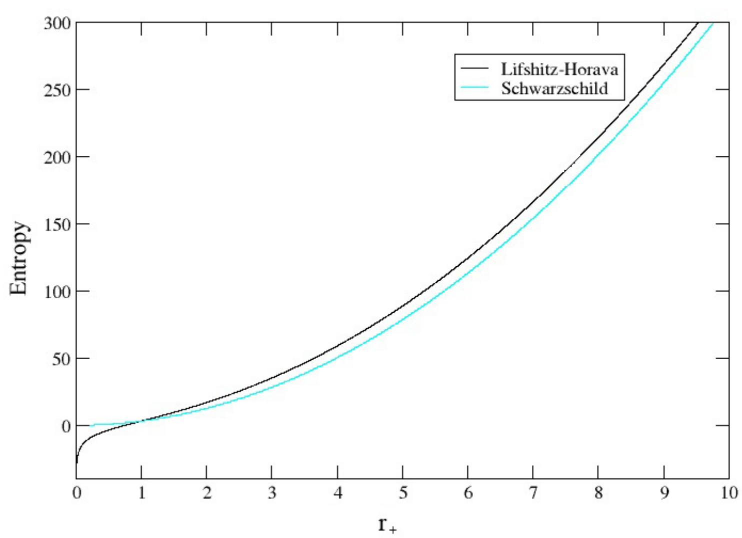

Figure 1: Entropy as a function of the radius of outer horizon with $\omega = 1$, i.e. $\alpha = \frac{1}{2}$.

On the other hand, Figure 2 shows the behavior of the entropy as a function of the horizon radius for different values of α for the Horava-Lifshitz black hole and for Schwarzschild's solution. When decreasing the value of α, i.e. increasing the value of ω, the entropy curve for Horava-Lifshitz black hole approaches Schwarzschild's entropy for small r_+. This behavior is better seen in Figure 3.

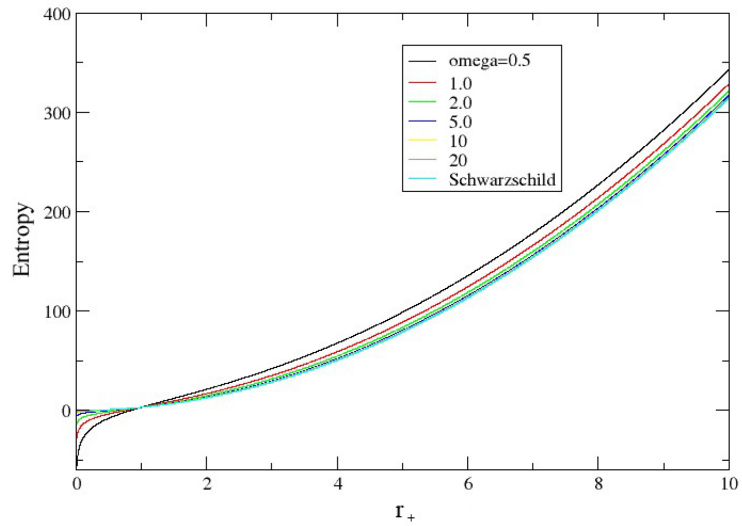

Figure 2: Entropy as a function of the radius of outer horizon for different values of ω.

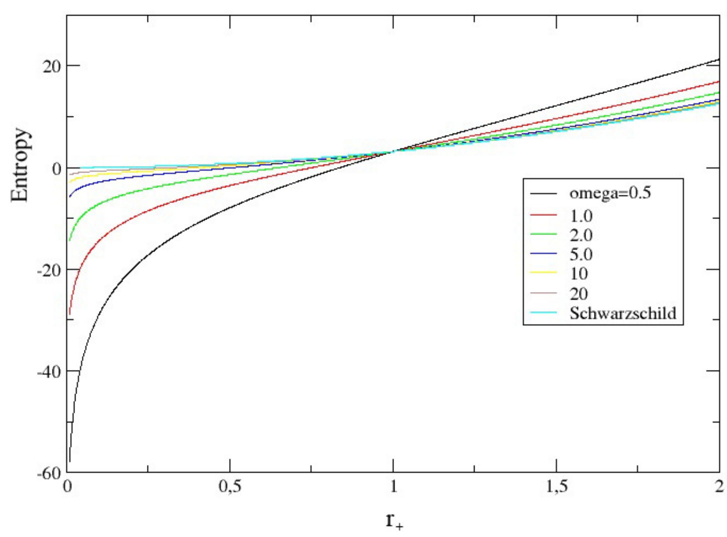

Figure 3: Zoom of Figure 2, Showing the entropy as a function of the radius of outer horizon for different values of ω.

IV. HEAT CAPACITY AND HAWKING-PAGE TRANSITION

The heat capacity can be calculated as

$$C = \frac{dM}{dT} = \frac{dM}{dr_+}\frac{dr_+}{dT}. \tag{29}$$

Using the temperature (21) and the mass function (23) we have

$$C(r_+) = -2\pi \frac{(r_+^2 + 2\alpha)^2 (r_+^2 - \alpha)}{r_+^4 - 5\alpha r_+^2 - 2\alpha^2}. \tag{30}$$

As is noted by Myung [5] an isolated black hole like Schwarzschild black hole is never in thermal equilibrium because it decays by the Hawking radiation. This can be seen from the negative value of its heat capacity by doing $\alpha = 0$ in (30), $C_S = -2\pi r_+^2$.

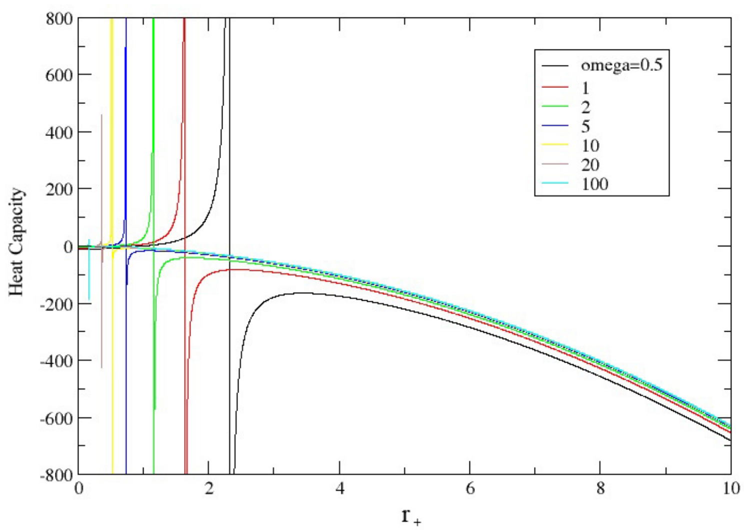

Figure 4: Heat capacity as a function of the radius of outer horizon for different values of ω.

In the case of the Horava-Lifshitz black hole, expression (30) shows that the heat capacity can be negative but also positive, depending on the value of the parameter α. In Figure 4 is easily seen that the heat capacity have positive values for different values of α. The value $r_+ = r_m$ at which the heat capacity blows is given by

$$r_m == \sqrt{\frac{5}{2}\sqrt{33}}\sqrt{\alpha}. \tag{31}$$

Black holes with $r_+ < r_m$ are local thermodynamically stable while those with $r_+ > r_m$ are unstable.

Finally, another interesting question is whether there exists the Hawking-Page phase transition associated with the Horava-Lifshitz black hole. In order to discuss the Hawking-Page transition, we have to calculate the Euclidean action or free energy for the black hole. The Euclidean action is related with the free energy by

$$I = \frac{1}{T}F, \tag{32}$$

where T is the temperature of the black hole and the free energy F is given by

$$F = M - TS. \tag{33}$$

Using equations (21), (23) and (28), we find

$$F = \frac{r_+^4 + 7\alpha r_+^2 + 4\alpha^2 + 4\alpha^2 \ln(r_+) - 4r_+^2 \alpha \ln(r_+)}{4r_+(r_+^2 + 2\alpha)}. \tag{34}$$

Note that the free energy is negative only for small enough horizon radius, which means that large black holes in Horava-Lifshitz gravity are thermodynamically unstable globally.

V. CONCLUSION

We studied the entropy of black holes in the deformed Horava-Lifshitz gravity with coupling constant λ. It has been shown that in the case $\lambda = 1$, the black hole resembles the Reissner-Norstrom black hole when it is noted that the geometric parameter $\alpha = \frac{1}{2\omega}$ in the horizon radius assumes a similar role as that the electric charge. The entropy of the Horava-Lifshitz black hole is calculated by assuming that the first law of thermodynamics is valid for this geometry. The obtained expression reduces to Schwarzschild's entropy in the limit $\alpha = 0$ but differs for other values. Finally we studied the heat capacity and Hawking-Page phase transition, to show that Black holes with $r_+ < r_m$ are globally thermodynamically stable, while large black holes are thermodynamically unstable globally.

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